Math 1131 Week 10 Worksheet

Name: ________________________

Discussion Section:_____________

**Solutions should show all of your work, not just a single final answer.**

# 4.7: Optimization Problems

1. A closed box (top, bottom, and all four sides) needs to be constructed to have a volume of $9\,\mathrm{m}^3$ and a base whose width is twice its length. See Figure 1.

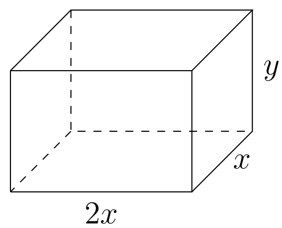


Figure 1: A box

Use calculus to determine the dimensions (length, width, height) of such a box that uses the least amount of material. Justify why your answer corresponds to a minimum, not a maximum.

2. We want to find the points on $y = x^2$ that are closest to $(0, 3)$.

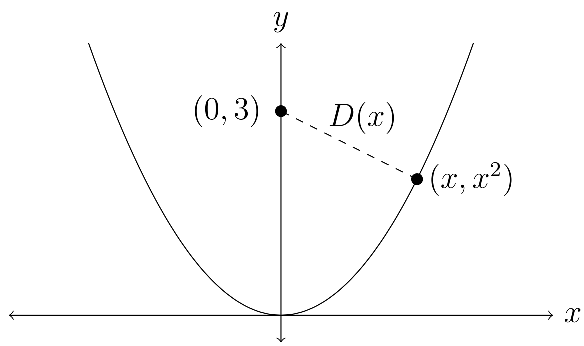


Figure 2: Distance to $(0, 3)$ on $y = x^2$.

(a) For each point $(x, x^2)$ on the parabola, find a formula for its distance to $(0, 3)$. Call this distance $D(x)$. (See Figure 2.)

(b) Let $f(x) = D(x)^2$, which is the *squared distance* between $(x, x^2)$ and $(0, 3)$. Finding where $D(x)$ is minimal is the same as finding where $f(x)$ is minimal. Determine all $x$ where $f(x)$ has an absolute minimum. The points $(x, x^2)$ for such $x$ are the closest points to $(0, 3)$ on $y = x^2$.

3. Three line segments of length 1 are joined together at endpoints to form a base and the legs of an isosceles trapezoid, as in Figure 3. Let $\theta$ in $(0, \pi/2)$ be the common angle measurement between the legs and the line passing through the base of length 1. We want to find the angle $\theta$ that maximizes the area of the trapezoid.

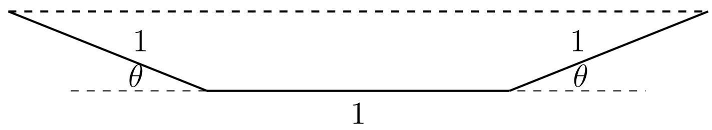


Figure 3: An isosceles trapezoid with base and legs of length 1.

(a) Compute the area $A(\theta)$ of the trapezoid. The general area formula for a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where $h$ is the height and $b_1$ and $b_2$ are the lengths of the bases. (Hint: Break up the trapezoid into a rectangle with two right triangles at both ends. Use trigonometry to compute the height and the length of the longer base in terms of $\theta$.)

(b) Find all solutions to $A'(\theta) = 0$ with $0 < \theta < \pi/2$. (Hint: Write $A'(\theta)$ entirely in terms of $\cos\theta$. The answer is *not* $\theta = \pi/4 = 45^o$.)

(c) Verify that the area $A(\theta)$ is a maximum, not a minimum, at the angle found in part (b) and compute this maximum area.

# 4.8: Newton's Method

4. Apply Newton's method to estimate the solution of $x^3 - x - 1 = 0$ by taking $x_1 = 1$ and finding the least $n$ such that $x_n$ and $x_{n+1}$ agree to three digits after the decimal point.

5. The number $\pi$ is a solution of $\sin x = 0$ close to 3 (see Figure 4). You will use Newton's method for $\sin x = 0$ to create numerical estimates for $\pi$.

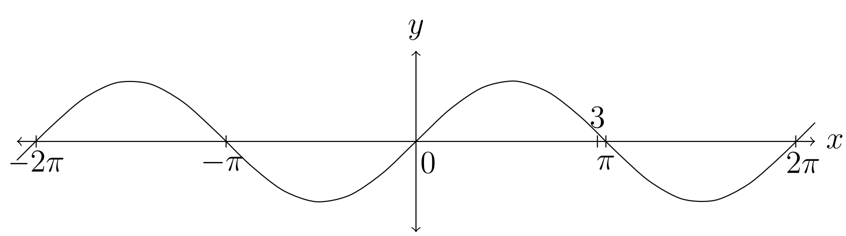


Figure 4: Graph of $y = \sin x$.

(a) Write out the recursion for Newton's method used to solve $\sin x = 0$.

(b) Using Newton's method for $\sin x = 0$ with $x_1 = 3$, find the first $n$ for which $x_n$ and $x_{n+1}$ agree to 5 digits after the decimal point. (Use radians, *not* degrees!)

(c) For the $n$ you found in part (b), to how many digits after the decimal point does $x_n$ actually agree with $\pi$?

6. In Figure 5 is the graph of $f(x) = \ln(x) - 1$ for $0 < x < 4$. It crosses the $x$-axis at $x = e$. You will use Newton's method for $f(x) = 0$ to create numerical estimates for $e$.

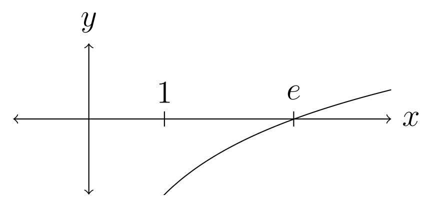


Figure 5: Graph of $y = \ln(x) - 1$.

(a) Using Newton's method for the equation $\ln(x) - 1 = 0$ with $x_1 = 1$, tabulate $x_n$ to find the first $n$ for which $x_n$ and $x_{n+1}$ agree to 5 digits after the decimal point.

(b) For the $n$ you found in part (a), to how many digits after the decimal point does $x_n$ actually agree with $e$?

Answers to Selected Problems:

1. The dimension are length $= x = \frac{3}{2}$, width $= 2x = 3$, and height $= \dfrac{9}{2(3/2)^2} = 2$.

2. (a) $D(x) = \sqrt{x^2 + (x^2 - 3)^2}$
   (b) $f(x)$ has absolute minima at $x = \pm\sqrt{5/2}$

3. (a) $A(\theta) = \sin\theta + (\sin\theta)(\cos\theta)$
   (b) $\theta = \pi/3$
   (c) Since $A''(\pi/3) < 0$, the maximum area is $\dfrac{3\sqrt{3}}{4}$

4. Use $n = 5$: $x \approx 1.3247\ldots$

5. (a) $x_{n+1} = x_n - \tan(x_n)$
   (b) $n = 3$: $x \approx 3.14159265330$
   (c) nine digits after the decimal point

6. (a) $n = 5$: $x_5 \approx 2.71828106$
   (b) The number $x_5$ agrees with $e$ to 6 digits after the decimal point, not just 5 digits.